Completions and simplicial complexes

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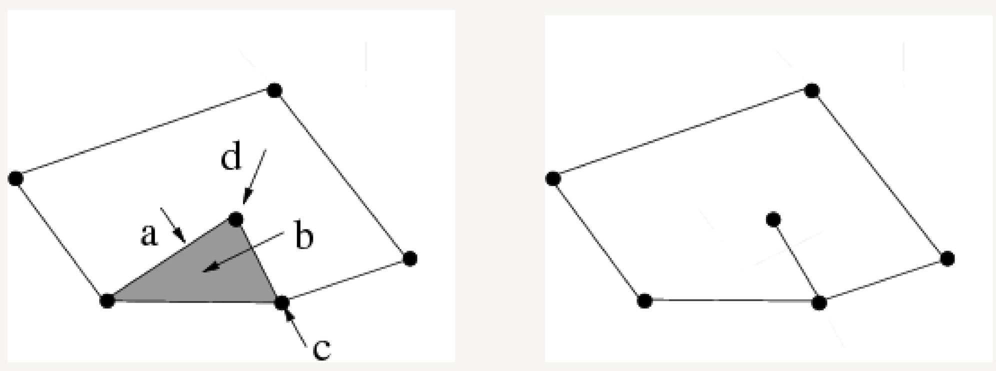
Abstract

We first introduce the notion of a **completion**. Completions are inductive properties which may be expressed in a declarative way and which may be combined. We show that completions may be used for describing structures or transformations which appear in combinatorial topology. We present two completions, $\langle \mathrm{Cup} \rangle$ and $\langle \mathrm{Cap} \rangle$, in order to define, in an axiomatic way, a remarkable collection of **acyclic complexes**. We give few basic properties of this collection. Then, we present a theorem which shows the equivalence between this collection and the collection made of all simply **contractible** simplicial complexes.

Topological spaces

When considering finite sets, a topological space is an **Alexandroff space**, *i.e.*, a topological space in which the intersection of any arbitrary family (not necessarily finite) of open sets is open.
There is a correspondance between Alexandroff spaces and **preorders** (binary relations that are reflexive and transitive).
A map between two preordered sets is monotone (*i.e.*, preserves the preorder relation) if and only if it is a continuous map between the corresponding Alexandroff spaces.



Two objects X (left) and Y (right).

Let us consider the (simplicial) objects X and Y. The object X is made of 6 vertices, 7 segments, and 1 triangle. A natural preorder $\leq$ between all these elements is the partial order corresponding to the relation of inclusion between sets. Thus we have $d \leq a$ and $a \leq b$.
We see that this is not possible to build a monotone map f between X and Y such that f is the identity on all elements of Y. For example, if we take $f(a) = c$, $f(b) = c$, we have $d \leq a$, but we have not $f(d) \leq f(a)$.
Thus, in the context of this construction, the classical axioms of topology fail to interpret Y as a continuous retraction of X.

Simplicial complexes

A **simplicial complex** is a finite family X composed of finite sets and such that, if $x \in X$ and $y \subseteq x$, then $y \in X$. We denote by $\mathbb{S}$ the collection of all simplicial complexes. Observe that $\emptyset \in \mathbb{S}$ and $\{\emptyset\} \in \mathbb{S}$.
An element of $X \in \mathbb{S}$ is a *face of* X. A *facet of* X is a face of X which is maximal for inclusion.
A complex $A \in \mathbb{S}$ is a **cell** if $A = \emptyset$ or if A has precisely one non-empty facet.
We write $\mathbb{C}$ for the collection of all cells.
Let $X \in \mathbb{S}$. We say that a face $x \in X$ is *free for* X if x is a proper face of exactly one face y of X, such a pair (x, y) is said to be a *free pair for* X. If (x, y) is a free pair for X, the complex $Y = X \setminus \{x, y\}$ is an elementary **collapse** of X.
Thus, the above object Y is an elementary collapse of X ((a, b) is a free pair).

Completions

In the sequel, the symbol $\mathbf{S}$ will denote an arbitrary collection. The symbol $\mathcal{K}$ will denote an arbitrary subcollection of $\mathbf{S}$, thus we have $\mathcal{K} \subseteq \mathbf{S}$. Let $\langle \mathrm{K} \rangle$ be a property which depends on $\mathcal{K}$. We say that a given collection $\mathbf{X} \subseteq \mathbf{S}$ satisfies $\langle \mathrm{K} \rangle$ if the property $\langle \mathrm{K} \rangle$ is true for $\mathcal{K} = \mathbf{X}$.
Let κ be a binary relation over $2^{\mathbf{S}}$ and $2^{\mathbf{S}}$, thus $\kappa \subseteq 2^{\mathbf{S}} \times 2^{\mathbf{S}}$. We say that κ is a *constructor (on* $\mathbf{S}$*)* if κ is *finitary*, which means that $\mathbf{F}$ is finite whenever $(\mathbf{F}, \mathbf{G}) \in \kappa$. If κ is a constructor on $\mathbf{S}$, we denote by $\langle \kappa \rangle$ the following property which is *the completion induced by* κ:
-> If $\mathbf{F} \subseteq \mathcal{K}$, then $\mathbf{G} \subseteq \mathcal{K}$ whenever $(\mathbf{F}, \mathbf{G}) \in \kappa$. $\langle \kappa \rangle$
The following theorem is a consequence of a fixed point property:

Theorem: *Let κ be a constructor on $\mathbf{S}$ and let $\mathbf{X} \subseteq \mathbf{S}$. There exists, under the subset ordering, a unique minimal collection which contains $\mathbf{X}$ and which satisfies $\langle \kappa \rangle$.*

We say that a property $\langle \mathrm{K} \rangle$ is a *completion* (on $\mathbf{S}$) if there exists a constructor κ such that $\langle \mathrm{K} \rangle$ is precisely the completion induced by κ.
If $\langle \mathrm{K} \rangle$ is a completion and if $\mathbf{X} \subseteq \mathbf{S}$, we write $\langle \mathbf{X}, \mathrm{K} \rangle$ for the unique minimal collection which contains $\mathbf{X}$ and which satisfies $\langle \mathrm{K} \rangle$.
Let $\langle \mathrm{K} \rangle$ and $\langle \mathrm{Q} \rangle$ be two completions. Then $\langle \mathrm{K} \rangle \wedge \langle \mathrm{Q} \rangle$ is a completion, the symbol $\wedge$ standing for the logical "and".
If $\mathbf{X} \subseteq \mathbf{S}$, the notation $\langle \mathbf{X}, \mathrm{K}, \mathrm{Q} \rangle$ stands for the smallest collection which contains $\mathbf{X}$ and which satisfies $\langle \mathrm{K} \rangle \wedge \langle \mathrm{Q} \rangle$.

Example: connectedness

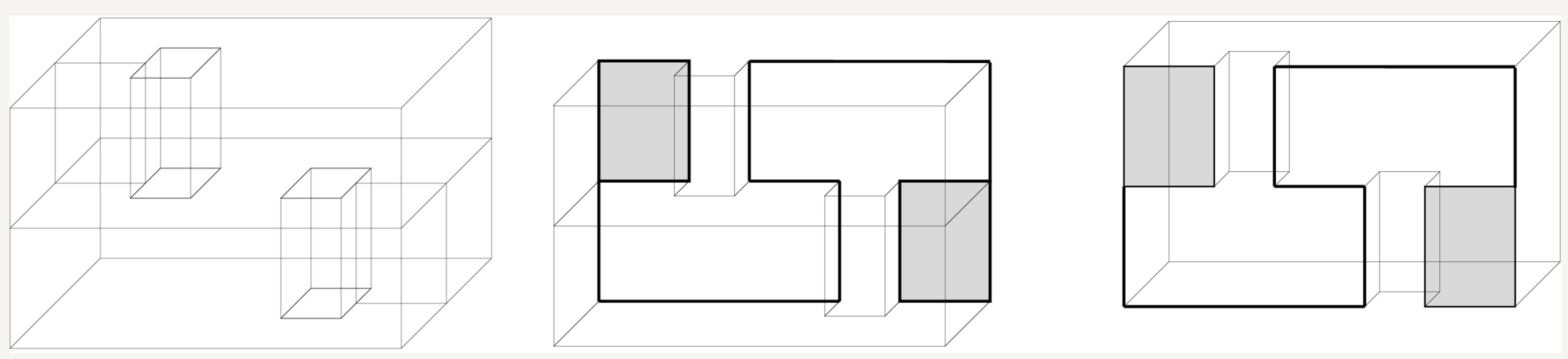
The family composed of all connected simplicial complexes may be defined by means of completions on $\mathbb{S}$. We define the completion $\langle \mathrm{PATH} \rangle$ as follows.
-> If $S \in \mathcal{K}$, then $S \cup C \in \mathcal{K}$ whenever $C \in \mathbb{C}$, and $S \cap C \neq \{\emptyset\}$. $\langle \mathrm{PATH} \rangle$
We set $\Pi = \langle \emptyset, \mathrm{PATH} \rangle$. We say that a complex $X \in \mathbb{S}$ is *connected* if $X \in \Pi$.
Observe that $\mathbb{C} \subseteq \Pi$ since, for any $C \in \mathbb{C}$, we have $C \cap \emptyset = \emptyset \neq \{\emptyset\}$.
It may be checked that this definition of a connected complex is equivalent to the classical definition based on paths.
Now, let us define the completion $\langle \Upsilon \rangle$ as follows.
-> If $S, T \in \mathcal{K}$, then $S \cup T \in \mathcal{K}$ whenever $S \cap T \neq \{\emptyset\}$. $\langle \Upsilon \rangle$
It may be verified that we have $\Pi = \langle \mathbb{C}, \Upsilon \rangle$. This last result shows that $\langle \Upsilon \rangle$ provides another way to generate the collection of all complexes which are in Π.

The Cup/Cap completions

We define the two completions $\langle \mathrm{Cup} \rangle$ and $\langle \mathrm{Cap} \rangle$:
-> If $S, T \in \mathcal{K}$, then $S \cup T \in \mathcal{K}$ whenever $S \cap T \in \mathcal{K}$. $\langle \mathbf{Cup} \rangle$
-> If $S, T \in \mathcal{K}$, then $S \cap T \in \mathcal{K}$ whenever $S \cup T \in \mathcal{K}$. $\langle \mathbf{Cap} \rangle$
We set $\mathbb{R} = \langle \mathbb{C}, \mathrm{Cup} \rangle$ and $\mathbb{D} = \langle \mathbb{C}, \mathrm{Cup}, \mathrm{Cap} \rangle$.
Each element of $\mathbb{R}$ is a *ramification* and each element of $\mathbb{D}$ is a *dendrite*.

The completions $\langle \mathrm{Cup} \rangle$ and $\langle \mathrm{Cap} \rangle$ may be seen as axioms which are used as "generators" for enumerating all the collection $\mathbb{D}$: we start from $\mathbb{C}$ and we inductively generate all elements of $\mathbb{D}$ by applying $\langle \mathrm{Cup} \rangle$ and $\langle \mathrm{Cap} \rangle$. In this sense, $\mathbb{D}$ may be seen as a "dynamic structure".

Now, let us introduce the two completions:
-> If $S \in \mathcal{K}$, then $T \in \mathcal{K}$ whenever S is an elementary collapse of T. $\langle \mathbf{Col} \rangle$
-> If $T \in \mathcal{K}$, then $S \in \mathcal{K}$ whenever S is an elementary collapse of T. $\langle -\mathbf{Col} \rangle$
We say that an element of $\langle \emptyset, \mathrm{COL} \rangle$ is *collapsible* and that an element of $\langle \emptyset, \mathrm{COL}, -\mathrm{COL} \rangle$ is *simply contractible*.
Remark: Any collapsible complex is a ramification and any ramification is a dendrite.



The Bing's house X (left), and two objects Y (middle) and Z (right).

The Bing's house with two rooms is a classical example of an object which is contractible but not collapsible. We see that the two complexes Y and Z are such that $X = Y \cup Z$. We also observe that Y, Z, and $Y \cap Z$ are collapsible, and therefore ramifications. Thus, the Bing's house X is a ramification.

The following theorem shows that $\mathbb{D}$ corresponds to a remarkable collection of acyclic complexes.

Theorem: *A simplicial complex is a dendrite if and only if it is simply contractible.*