Smooth 2D Coordinates on Discrete Surfaces

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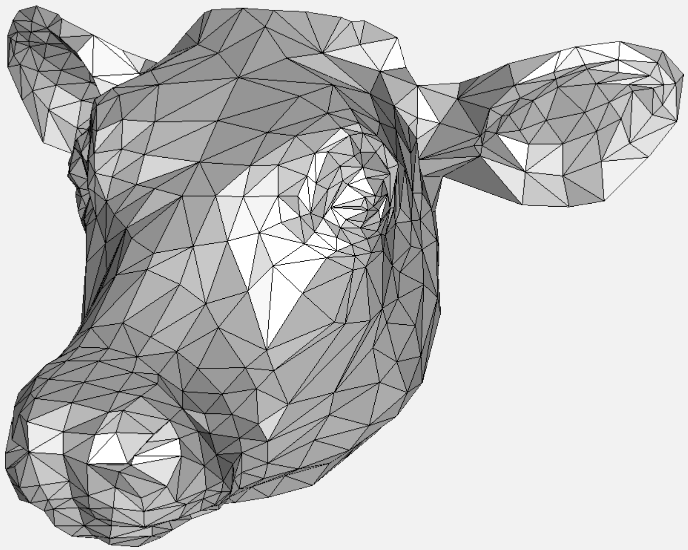
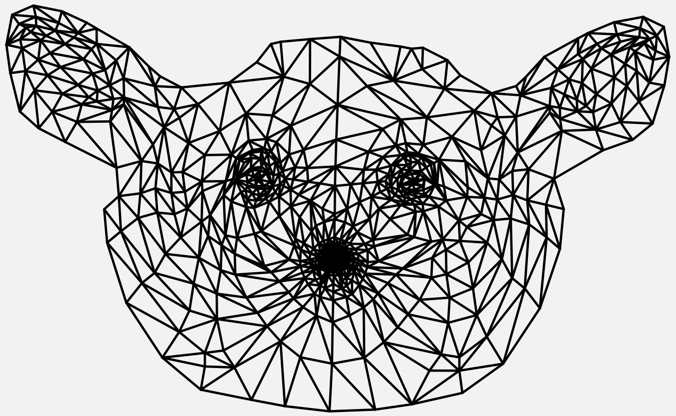
DGCI 2011, April 8th

Main problem

- Parameterized surface : $(s,t) \in \mathbf{R}^2 \mapsto \big(x(s,t), y(s,t), z(s,t)\big) \in \mathcal{S}$
- Parameterization : $(x,y,z) \in \mathcal{S} \mapsto \big(s(x,y,z), t(x,y,z)\big) \in \mathbf{R}^2$
- Conformal $\iff$ preserving angles



3D mesh

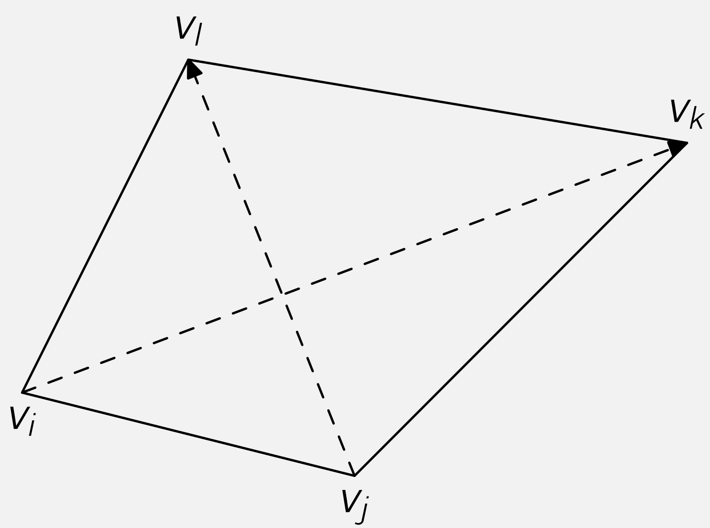


2D parameterization

Outline

1 – Discrete conformal parameterizations

1.1 – Quadrangular meshes



- Conformal structure

$$\rightarrow \rho = \frac{v_l - v_j}{i \times (v_k - v_i)} \in \mathbf{C}$$

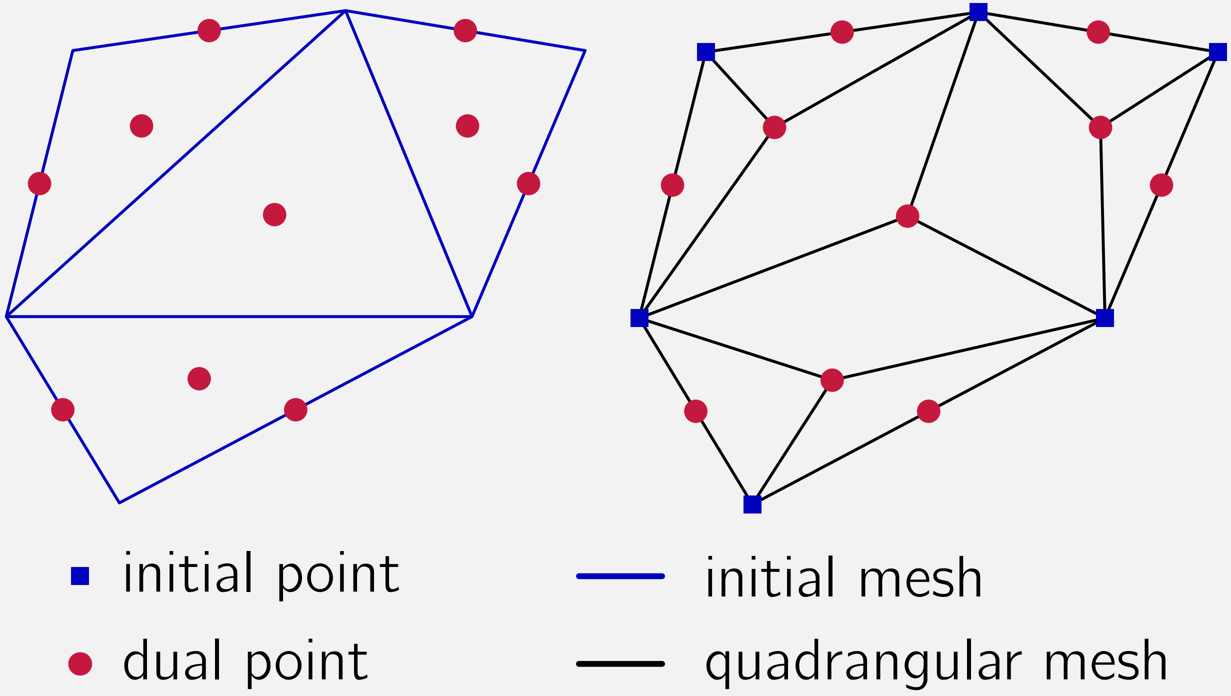
- Parameterization : $v_i \in \mathcal{S} \mapsto v_i' \in \mathbf{C}$
- Linear system

$$v_l' - v_j' = i\rho(v_k' - v_i')$$

Mercat, C.: Discrete Riemann surfaces and the Ising model. Communications in Mathematical Physics 218(1), 177-216 (2001)

1.2 – Triangular meshes

- Construction of a quadrangular mesh

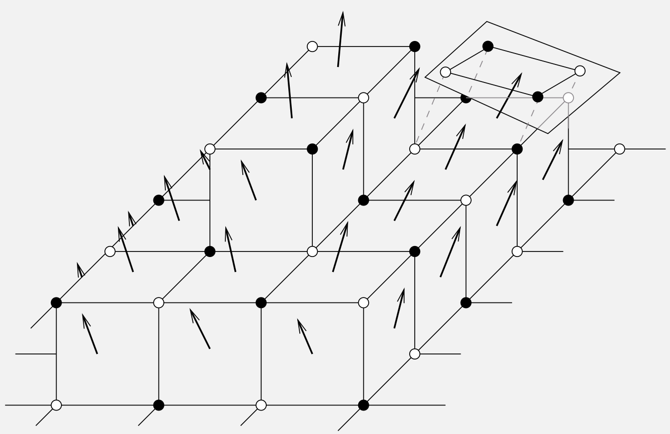


- Parameterization of the quadrangular mesh.

1.3 – Digital surfaces

- Face = Surfel = Square
  $\rightarrow$ $\rho = 1$

- Estimation of (continuous) normales



- Projection of the surfel on the tangent plane
  $\rightarrow$ parallelogram
- *Definition :*
  ρ of a surfel = ρ of the parallelogram.

Fourey, S., Malgouyres, R.: Normals estimation for digital surfaces based on convolutions. Computers & Graphics 33(1), 210 (2009).

Mercat, C.: Discrete complex structure on surfel surfaces. In: Discrete Geometry for Computer Imagery. pp. 153164. Springer (2008)

2 – Boundary conditions

2.1 – Riemann mapping theorem

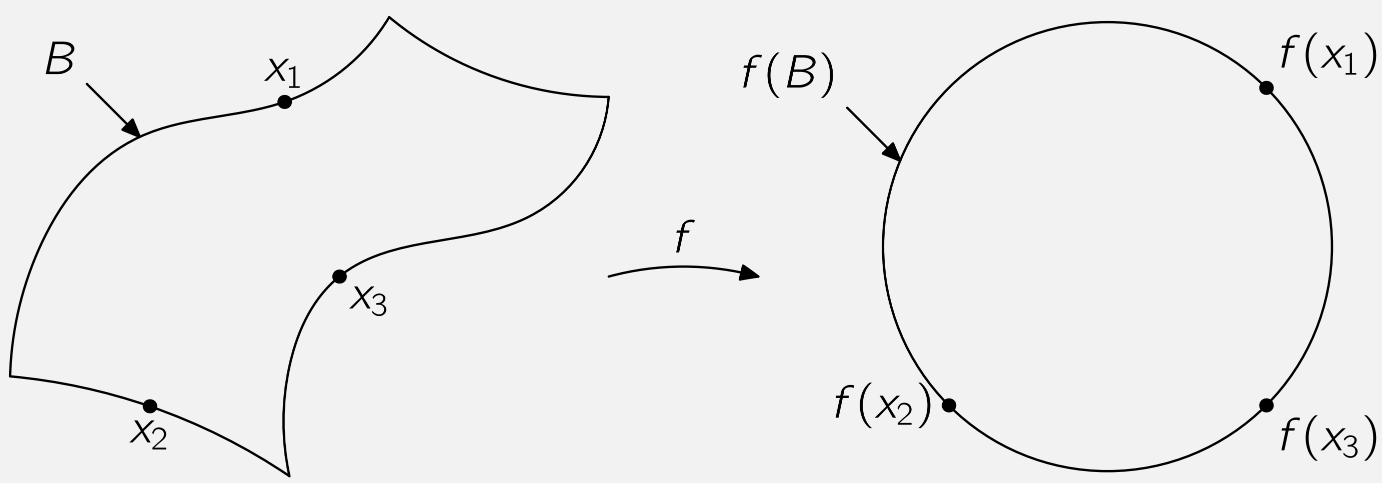
- Riemann mapping theorem

  surface homeomorphic to a disk $\Rightarrow$ has a conformal parametrisation

- Uniqueness if:

  $\longrightarrow$ the boundary is mapped on the unit circle

  $\longrightarrow$ the images of 3 boundary points are fixed



2.2 – Discrete version

- Notations : n_v vertices n_e edges n_f faces
  n_b boundary points

- $n_b + 2$ degrees of freedom
  $\longrightarrow$ *Hint:*

$$\begin{cases} 4n_f = 2n_e - n_b \\ 1 = \text{Euler characteristic of the disk} \\ \quad = n_f - n_e + n_v \end{cases}$$

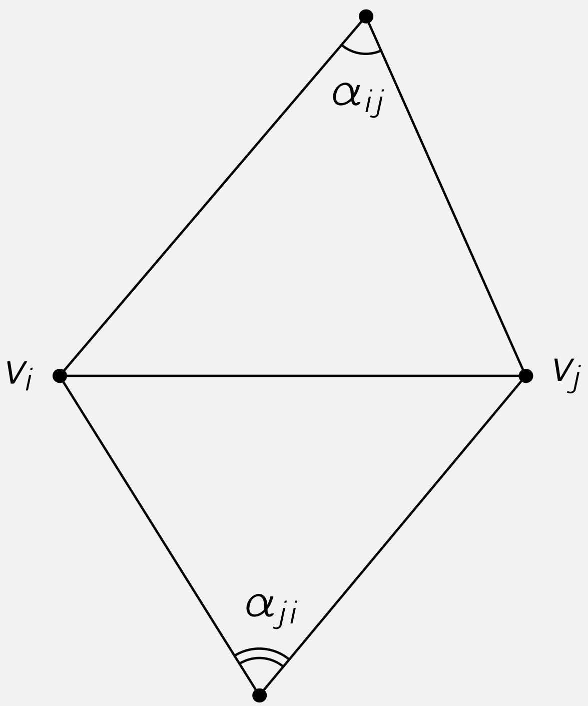
- Discrete version of the Riemann mapping theorem
  $\longrightarrow$ we send the boundary onto the disk
  $\longrightarrow$ we fix two (almost three) boundary points

2.3 – Generalization of the cotan method for triangular meshes

- Method
  - $\longrightarrow$ they fix the images of the boundary points
  - $\longrightarrow$ for each interior vertex v_i,

$$\sum_{j\ :\ v_j\ \text{neighbour of}\ v_i} \left(\cot\alpha_{ij} + \cot\alpha_{ji}\right)(v'_j - v'_i) = 0$$

  - $\longrightarrow$ linear system



- *Remark:* similar to the relaxation of a network of springs.

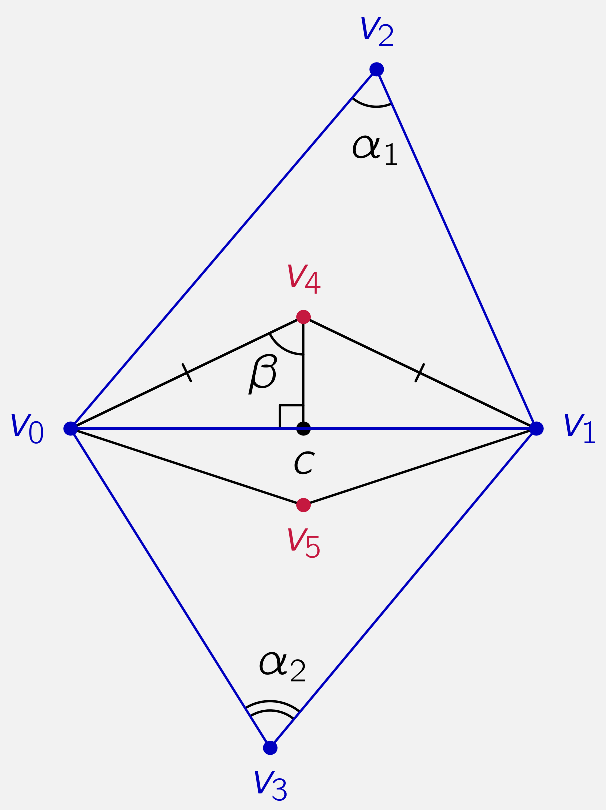
Pinkall, U., Polthier, K.: Computing discrete minimal surfaces and their conjugates. Experimental mathematics 2(1), 1536 (1993).

- Dual points
  - $\longrightarrow$ circumcenters of the triangles
  - $\longrightarrow$ middle of the boundary edges

- Boundary constraints
  - $\longrightarrow$ fix initial boundary points as with the *cotan* method
  - $\longrightarrow$ fix one the dual boundary points

- Same parameterization as with the *cotan* method.
  - $\longrightarrow$ *Hint:*

    $\rho(v_0, v_5, v_1, v_4) = \cot\alpha_1 + \cot\alpha_2$



3 – Practical computation

3.1 – Minimizing energies

- Conformal energy

$$H = \sum \left|(v'_l - v'_j) - \rho_f \left(v'_k - v'_i\right)\right|^2,$$

sum over all the faces $f = (v_i, v_j, v_k, v_l)$.

- Boundary energy

$$C = \sum \left(|v'_i|^2 - 1\right)^2$$

sum over the boundary vertices except the fixed ones.

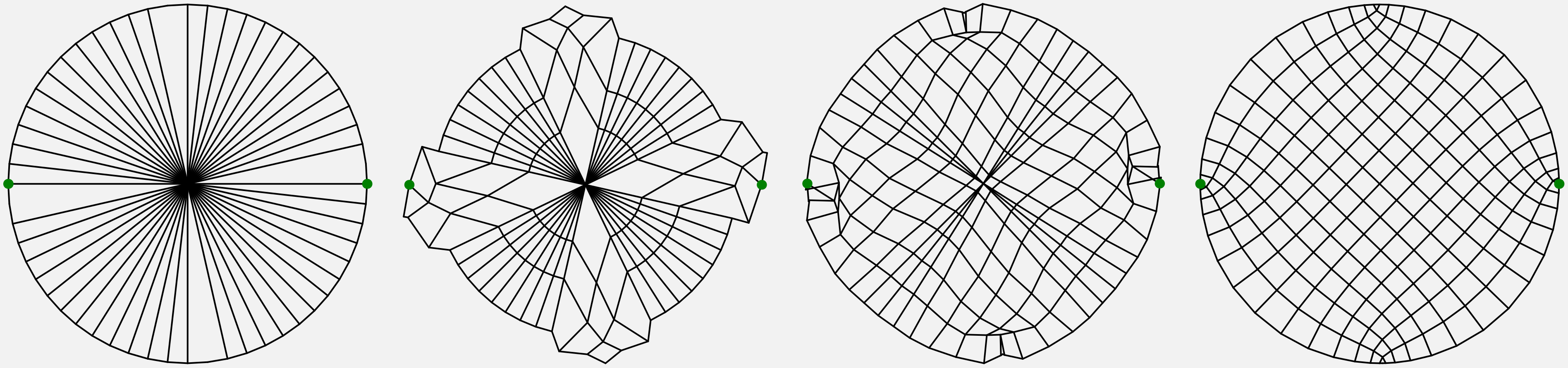
- We look for the v'_i coordinates that minimize

$$E = \alpha H + \beta C$$

using a Newton Method (BFGS).

3.2 – Initialization

- Boundary points on the unit circle.
- Interior points in 0.
- Fixed points as far as possible from each other.



3.3 – Preserving lengths and areas

- Preserving lengths

$$L = \sum \left(|v'_i - v'_j|^2 - \|v_i - v_j\|^2 \right)^2,$$

sum over all the edges $[v_i, v_j]$ (or only boundary ones).

- Preserving areas

$$A = \sum \left(\mathrm{Im}(v'_k - v'_i)\overline{(v'_l - v'_j)} - \|(v_l - v_i) \wedge (v_k - v_i)\| - \|(v_k - v_i) \wedge (v_j - v_i)\| \right)^2$$

sum over all the faces (v_i, v_j, v_k, v_l).

- 2 steps algorithm

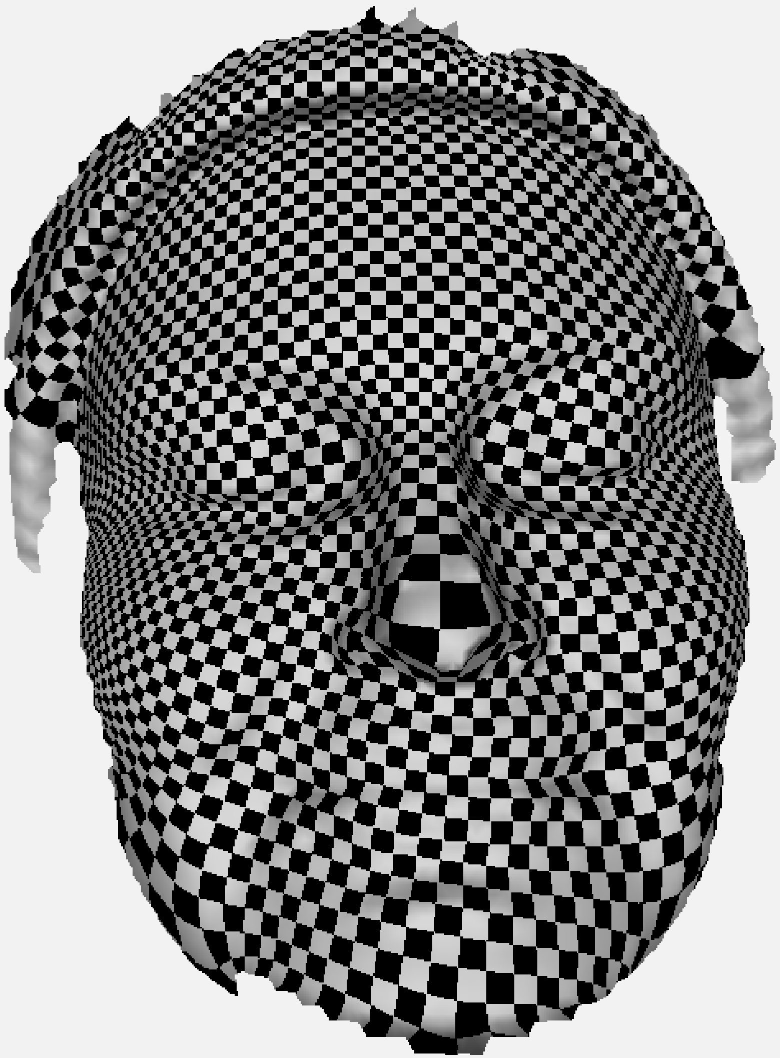
→ minimization of H with fixed boundary

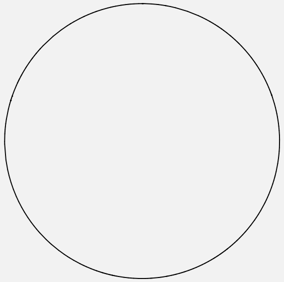
→ use this minimum as initial condition to minimize

$E = \alpha H + \beta L + \gamma A + \cdots$.

4 – Numerical results

4.1 – Comparison of energies



- Energy : $E = H + C$
- Boundary : 
- Distortions :

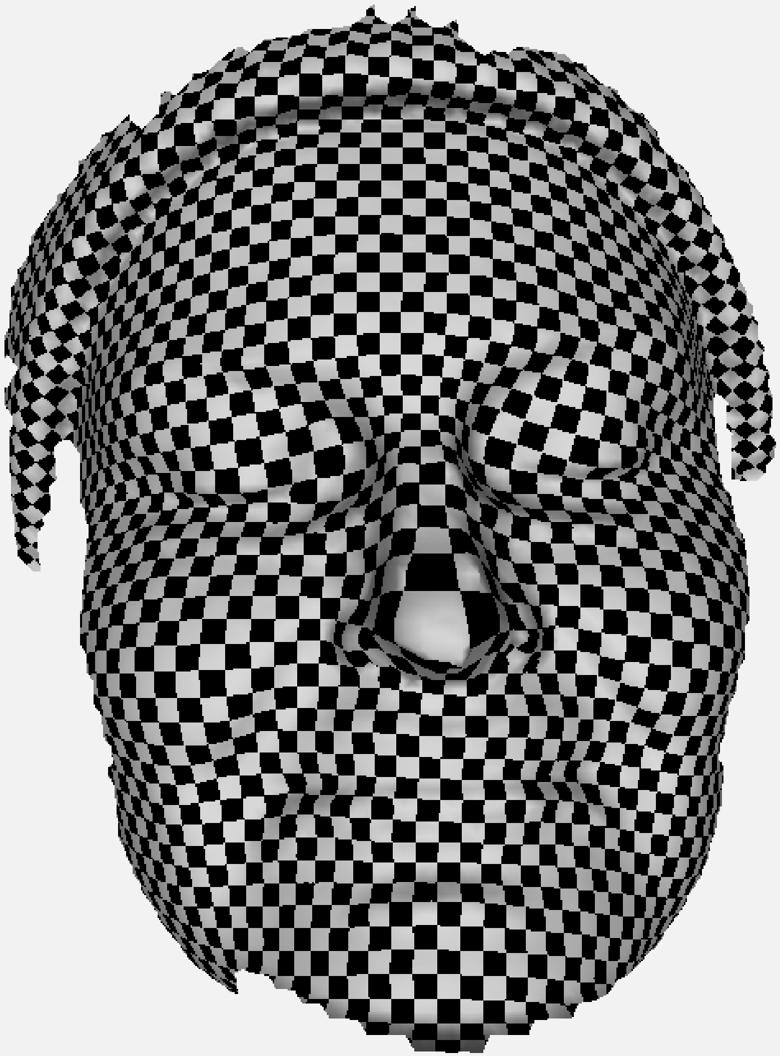
$\longrightarrow$ angles: 0.01 (radian)

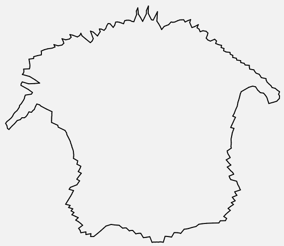
$\longrightarrow$ areas: 0.37 (ratio, should be 1)

$\longrightarrow$ lengths: 0.58 (ratio)

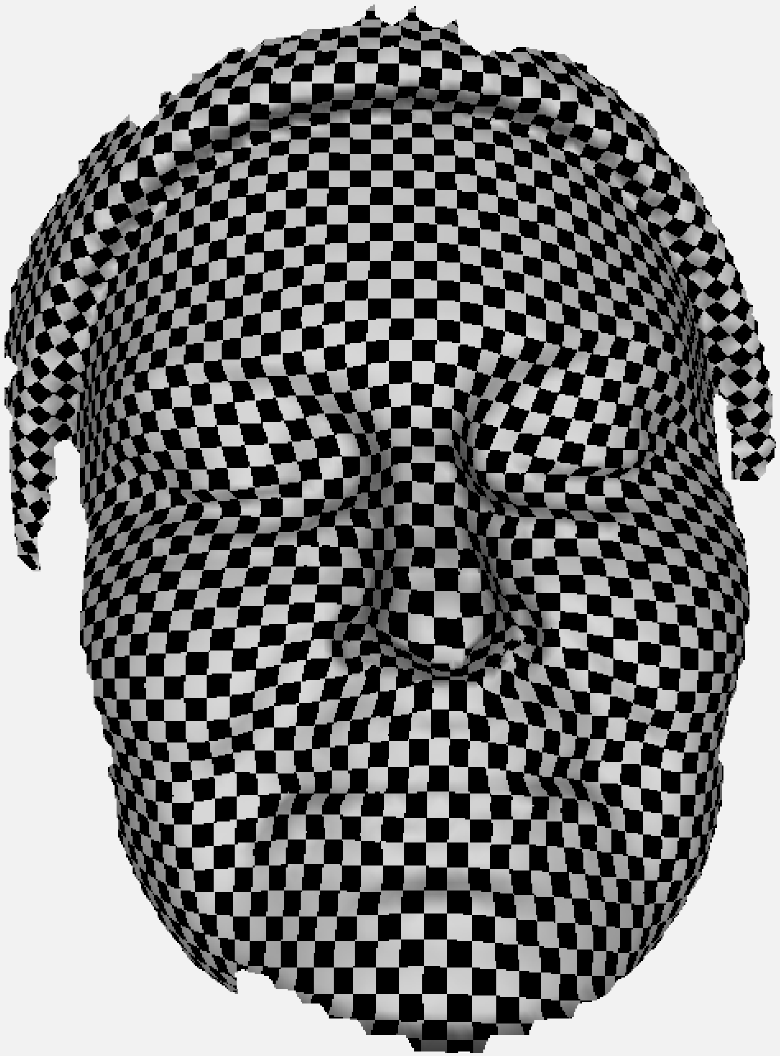
$\longrightarrow$ conformal

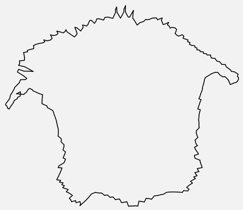
very unnatural boundary



- Energy : $E = H + L$ (boundary only)
- Boundary : 
- Distortions :
  - → angles: 0.01 (radian)
  - → areas: 0.88 (ratio, should be 1)
  - → lengths: 0.93 (ratio)

  → conformal,
  natural boundary,
  area distortion

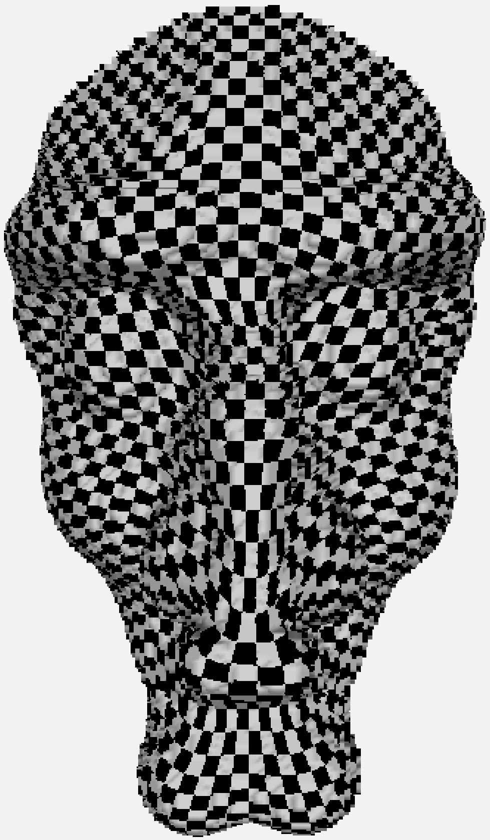


- Energy : $E = H + A$
- Boundary : 
- Distortions :
  - $\longrightarrow$ angles: 0.08
  - $\longrightarrow$ areas: 0.98 (ratio, should be 1)
  - $\longrightarrow$ lengths: 0.96 (ratio)

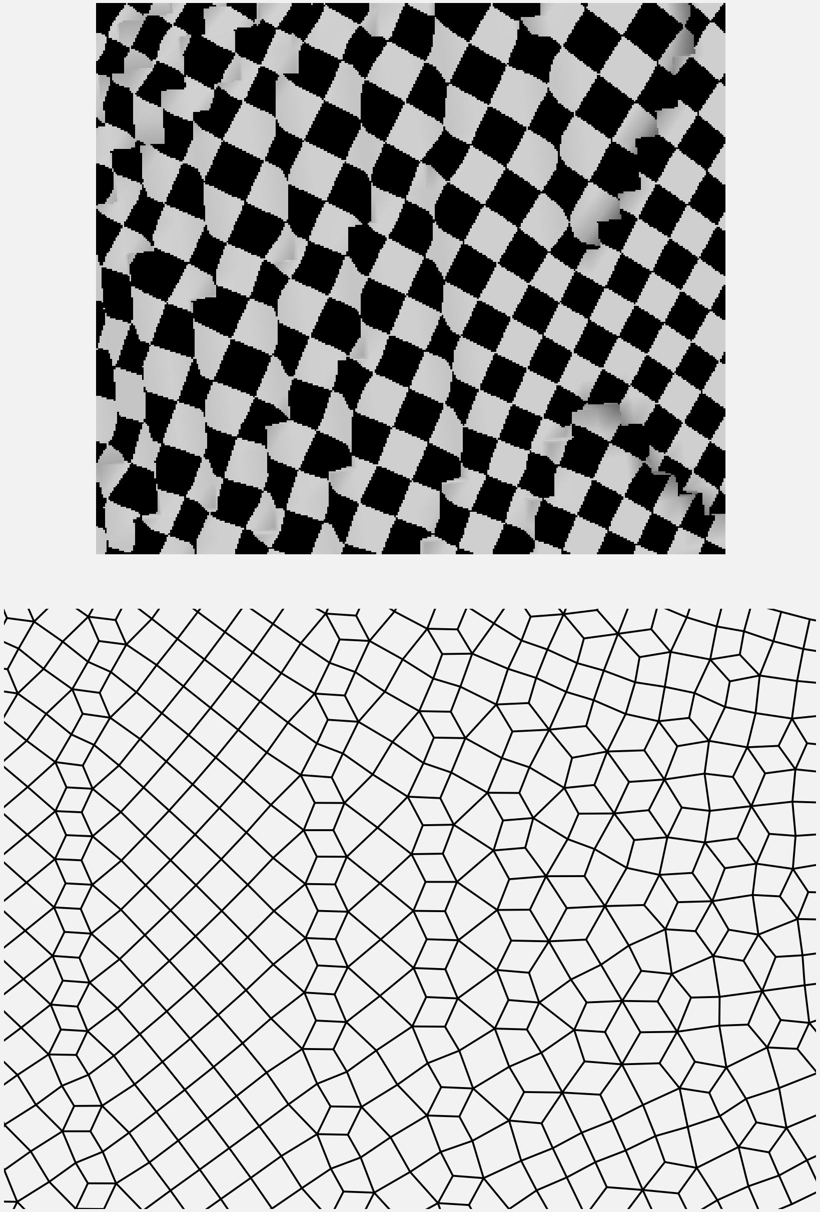
  $\longrightarrow$ natural boundary,
  good texture mapping,
  quasi-conformal

4.2 – Comparison with classical methods for digital surfaces

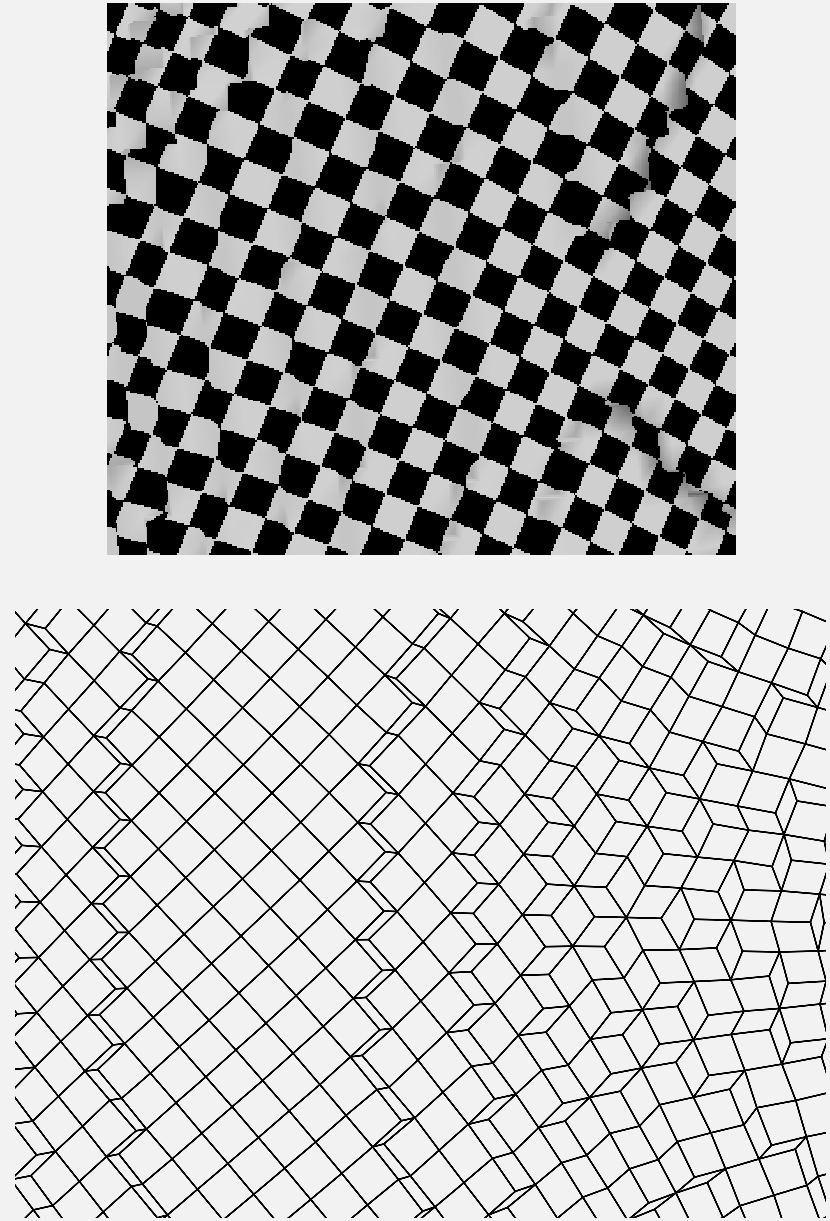
$E = H + L$



$E = H + A$



ABF



Voxel method, $E = H + 0.01A$

Conclusion

- A method that can be applied to
  - → quadrangular meshes
  - → triangular meshes
  - → digital surfaces

- A discrete version of the Riemann mapping theorem

- A flexible method: can preserve more or less
  - → shapes
  - → metric
  - → boundaries
  - → . . .