Our ultimate purpose:

To extract (co)homological information of a 3D model.
For this aim, first we need to compute representative (co)cycles of (co)homology generators of dimension 1 in the model.

Computing loops on the surface that wraps around their ‘handles’ and ‘tunnels’.

Applications:
- Feature detection
- Topological simplification
- ...

All the computations are carried out over a connected closed surface in $\mathbb{R}^3$.

Can we start from a similar scenario if we consider the cubical complex associated to a 3D digital picture?

Our answer: Well-composed cell complexes
Well-composed images enjoy important topological and geometric properties:

1. There is only one type of connected component.

2. Some algorithms used in computer vision, computer graphics and image processing are simpler.
Introduction

- Well-composed images enjoy important topological and geometric properties:

  3. Thinning algorithms can be simplified and naturally made parallel if the input image is well-composed.

  4. Some algorithms for computing surface curvature or extracting adaptive triangulated surfaces assume that the input image is well-composed.
There are several methods for turning binary digital images that are not well-composed into well-composed ones:

...but these methods “destroy the topology”.
Our goal in this paper:

To “transform” the cubical complex induced by a 3D binary digital picture into a homotopy equivalent cell complex, whose boundary is made up by 2-manifolds:

WELL–COMPOSED CELL COMPLEX.
I = (\mathbb{Z}^3, B): set of unit cubes (voxels) centered at the points of B together with all the faces.

Example: I = (\mathbb{Z}^3, B), B = \{(0,0,0), (0,1,0), (0,0,1), (1,0,1)\}

B = foreground
B^c = \mathbb{Z}^3 \setminus B = background
Cubical complex:

\{ \text{Voxels} \} \leftrightarrow \{ \text{3D cubes in } \mathbb{R}^3 \} \downarrow

Combinatorial structure: CUBICAL COMPLEX

- 0-cells = vertices
- 1-cells = edges
- 2-cells = squared faces
- 3-cells = cubes

Voxel
Cubical complex:

\{ \text{Voxels} \} \xleftrightarrow{} \{ \text{3D cubes in } \mathbb{R}^3 \} \xrightarrow{}

Combinatorial structure: CUBICAL COMPLEX

0-cells notation:

\[ p_i = (i \pm \frac{1}{2}, j \pm \frac{1}{2}, k \pm \frac{1}{2}) \]
Cubical complex:

{ Voxels } \rightarrow \{ 3D \text{ cubes in } \mathbb{R}^3 \}

Combinatorial structure: CUBICAL COMPLEX

1-cells notation:

\[ a_i = (i, j \pm \frac{1}{2}, k \pm \frac{1}{2}) \]
\[ a_i = (i \pm \frac{1}{2}, j, k \pm \frac{1}{2}) \]
\[ a_i = (i \pm \frac{1}{2}, j \pm \frac{1}{2}, k) \]
Cubical complex:

\{ \text{Voxels} \} \leftrightarrow \{ \text{3D cubes in } \mathbb{R}^3 \}

Combinatorial structure: **CUBICAL COMPLEX**

2-cells notation:

\[ c_l = (i \pm \frac{1}{2}, j, k) \]
\[ c_l = (i, j \pm \frac{1}{2}, k) \]
\[ c_l = (i, j, k \pm \frac{1}{2}) \]
Cubical complex:

\{ \text{Voxels} \} \leftrightarrow \{ \text{3D cubes in } \mathbb{R}^3 \}

Combinatorial structure: CUBICAL COMPLEX
Well-composed images:

- $I = (\mathbb{Z}^3, B)$ is a well-composed image if the boundary of the cubical complex associated, $\partial Q(I)$, is a 2D-manifold.

- [Latecki97] A 3D binary digital image is well-composed iff the configurations $C_1$, $C_2$ and $C_3$ do not occur in $Q(I)$. 
This point has not a neighborhood homeomorphic to $\mathbb{R}^2$. 

Well-composed images:
Well-composed images:

Critical configurations within a block of eight cubes
A 3D digital image is not generally a well-composed image

Cubical complex $Q(I)$

Homotopy equivalent

Cell complex $K(I)$ such that $\partial K(I)$ is composed by 2D-manifolds: a well-composed cell complex.
Key idea: to create a true face adjacency to avoid the critical configurations

Cubical complex $Q(I)$

Homotopy equivalent

Cell complex $K(I)$ such that $\partial K(I)$ is composed by 2D-manifolds: a well-composed cell complex.
From a cubical complex to a well-composed cell complex:

**INPUT**
Cubical complex $Q(I)$

**OUTPUT**
Well-composed cell complex $K(I)$
So, we need “more space” to add new cells:

Adjacency = face relation between cells.
From a cubical complex to a well-composed cell complex:

1. Label critical edges and critical vertices of $Q(I)$.
2. Repair critical edges of $Q(I)$.
3. Repair critical vertices of $Q(I)$.

Cubical complex $Q(I)$

Homotopy equivalent

Cell complex $K(I)$ such that $\partial K(I)$ is composed by 2D-manifolds: a well-composed cell complex.
Step 1: Label critical edges and critical vertices of $Q(I)$.

$A :=$ set of critical edges of $Q(I)$
$V^i :=$ set of critical vertices of $Q(I)$
From a cubical complex to a well-composed cell complex:

Step 1: Label critical edges and critical vertices of $Q(I)$.

Step 2: Repair critical edges of $Q(I)$. 
Computing well-composed cell complexes

Step 1: Label critical edges and vertices.

Step 2: Repair critical edges of Q(I).
Computing well-composed cell complexes efficiently

Step 1: Label critical edges and vertices.

Step 2: Repair critical edges of Q(I).

Adjacency = face relation between cells.
From a cubical complex to a well-composed cell complex:

Step 1: Label critical edges and critical vertices of $Q(I)$.

Step 2: Repair critical edges of $Q(I)$.

Step 3: Repair critical vertices of $Q(I)$.

Example 1
Step 1: Label critical edges and critical vertices of $Q(I)$.

Step 2: Repair critical edges of $Q(I)$.

Step 3: Repair critical vertices of $Q(I)$.

Computing well-composed cell complexes
From a cubical complex to a well-composed cell complex:

Step 1: Label critical edges and critical vertices of $Q(I)$.

Step 2: Repair critical edges of $Q(I)$.

Step 3: Repair critical vertices of $Q(I)$.
Computing well-composed cell complexes

Step 1: Label critical edges and critical vertices of $Q(I)$.

Step 2: Repair critical edges of $Q(I)$.

Step 3: Repair critical vertices of $Q(I)$.
From a cubical complex to a well-composed cell complex:

Step 1: Label critical edges and critical vertices of $Q(I)$.

Step 2: Repair critical edges of $Q(I)$.

Step 3: Repair critical vertices of $Q(I)$.
From a cubical complex to a well-composed cell complex:

Step 1: Label critical edges and critical vertices of Q(I).

Step 2: Repair critical edges of Q(I).

Step 3: Repair critical vertices of Q(I).
Aims:
- To compute the homology of the foreground image as well as the background by computing the homology of the boundary surface;
- Geometrically control the representative (co)-cycles of homology generators;
- Deal with other 3D digital images: (6,26), (18,6) or (6,18) 3D images.
Thanks for your attention!

Questions