Metric bases for polyhedral gauges

Fabien Rebatel, Édouard Thiel

Laboratoire d'Informatique Fondamentale de Marseille
Université de la Méditerranée

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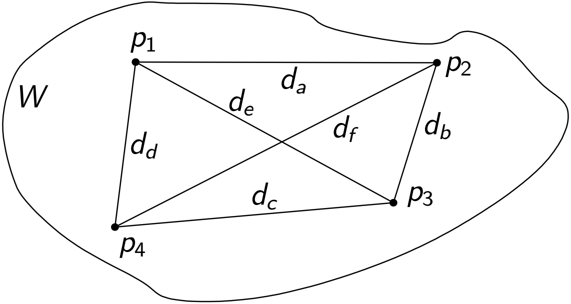
Metric bases



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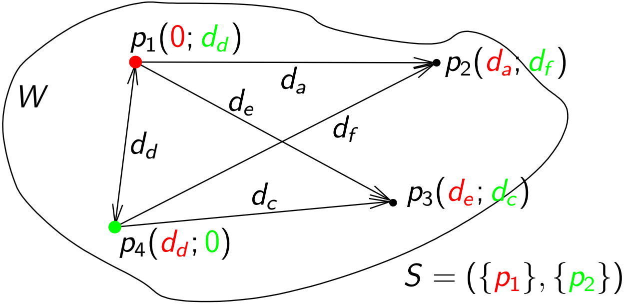
Metric bases



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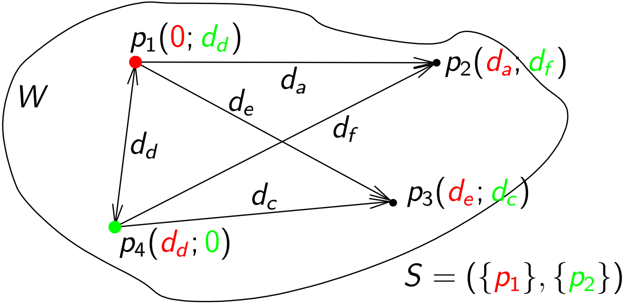

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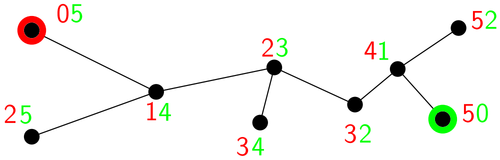
A subset $S \subseteq W$ is a **resolving set** for W if $d(x, p) = d(y, p)$ for all $p \in S$ implies $x = y$.
A **metric basis** is a resolving set of minimal cardinality $\#S$, named the **metric dimension** of (W, d).

In graph theory

F.Harary and R.Melter in 1976 :

- metric dimension for paths, complete graphs, and some other classes of graphs,
- an algorithm for computing metric bases in trees.



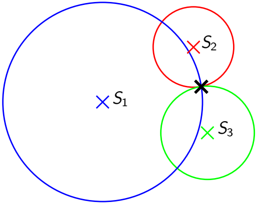
S.Khuller, B.Raghavachari and A.Rosenfeld in 1996 :

- an efficient algorithm for computing metric bases for trees (in linear time),
- finding metric dimension for an arbitrary graph is NP-hard,
- approximated algorithm factor $O(\log n)$ for arbitrary graphs.

In continuous geometry

R.Melter and I.Tomescu in 1986 :

- 3 non collinear points form a metric basis for the Euclidean distance in the plane,
- no finite basis for d_1 and d_∞ in the plane,
- the dimension for d_1 in a rectangle is 2,
- the dimension for d_∞ in a square is 3.



G.Chartrand, P.Zhang and G.Salehi in 1998, 2000 and 2001 :

- Partition dimension problem.
- Forcing subset problem.

Our aim

Study **metric basis** for usual discrete distances (**chamfer norms**).

First step

Study **polyhedral gauges** which are the $\mathbb{R}^n$ generalization of chamfer Norms.

Definition

Given a convex $\mathcal{C}$ containing the origin O in its interior, a **gauge** for $\mathcal{C}$ is the function $\gamma_{\mathcal{C}}(x)$ defined by the minimum positive scale factor λ, necessary for having $x \in \lambda\mathcal{C}$.

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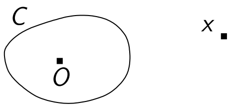

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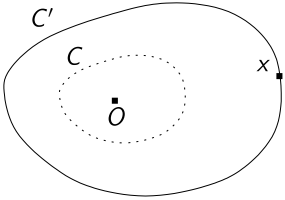

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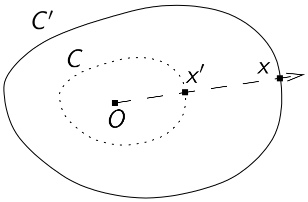

In the continuous space $\mathbb{R}^n$

R.Melter and I.Tomescu (1986)

There are no metric basis for d_1 and d_∞ in the plane.

Theorem

There are no metric bases for polyhedral gauges in $\mathbb{R}^n$.

Proof Idea

1. In any polyhedral cone, it always exists at least two points which have the same distance to the origin.
2. The intersection between a finite number of similar polyhedral cones is always an unbounded and non empty polyhedron.

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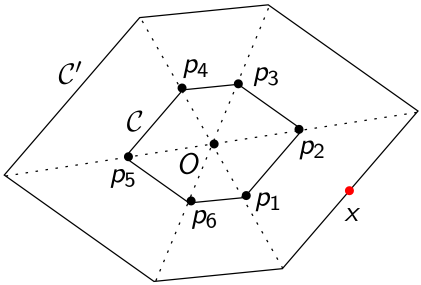
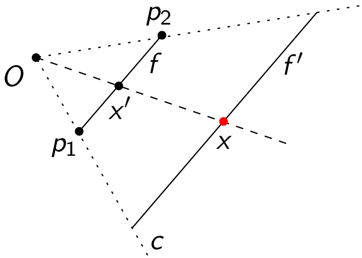
Proof idea 1 - Influence cones in polyhedral gauges

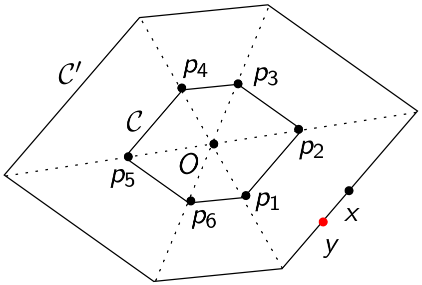


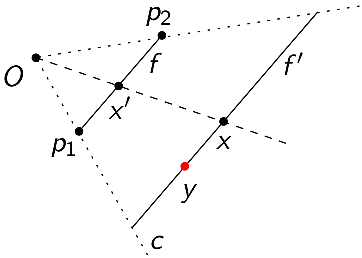


Splitting the space into cones

Measuring distances to a point x in a polyhedral cone (O, p_1, p_2).

Proof idea 1 - Influence cones in polyhedral gauges

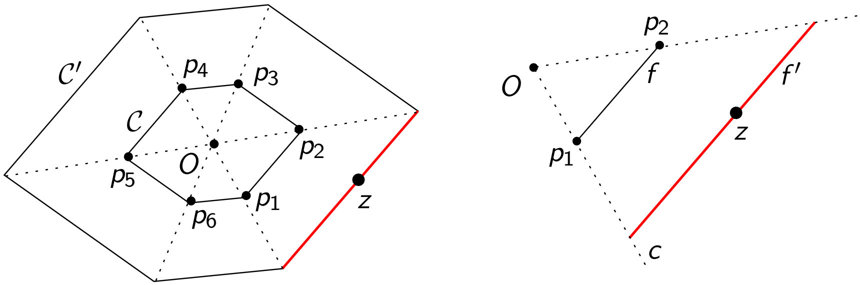




Splitting the space into cones

If an other point y belongs to f' then $d_{\mathcal{C}}(O, x) = d_{\mathcal{C}}(O, y)$

Proof idea 1 - Influence cones in polyhedral gauges



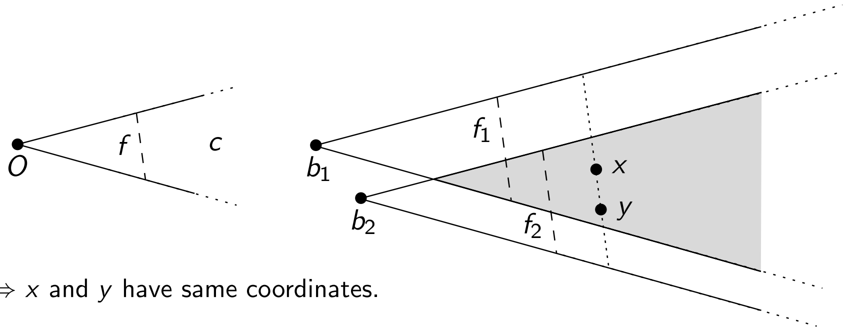
Splitting the space into cones

Every points z of f' have the same distance $d_{\mathcal{C}}(O, z)$.

Proof idea 2 - Intersections of cones

Lemma

Intersection between two translated polyhedral cones is an unbounded and non empty area.

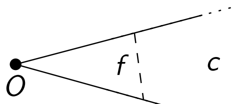


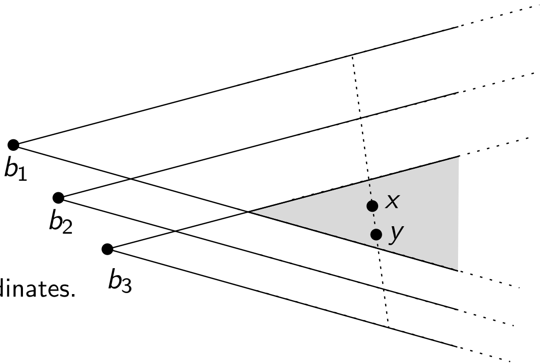
$\Longrightarrow$ x and y have same coordinates.

Proof idea 2 - Intersections of cones

Lemma extended

The previous lemma remain valid for any number of cones.





$\Longrightarrow$ x and y have same coordinates.

In the discrete space $\mathbb{Z}^n$

Does our theorem remains valid in $\mathbb{Z}^n$?

- Yes, if the gauge is **rational**;
- No, in the other cases.

Why ?

in $\mathbb{Z}^n$ a line may intersect a single point.

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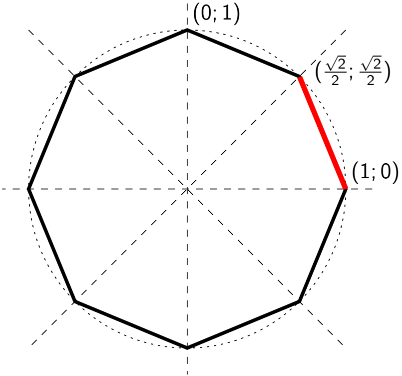
For instance, in $\mathbb{Z}^2$, if a line $L : y = Ax + B$ intersects two points $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$, then $A = \frac{(y_2 - y_1)}{(x_2 - x_1)}$. So $A \in \mathbb{Q}$.

Example of non-rational gauge



The slope of the red facet is $\frac{1}{\frac{\sqrt{2}}{2}-1} \notin \mathbb{Q}$.

Non-rational faces

Each facet of this gauge has a non-rational slope.

Chamfer distances

A **Chamfer mask** $\mathcal{M}$ in $\mathbb{Z}^n$ is a central-symmetric set $\mathcal{M} = \{(\vec{v_i}, w_i) \in \mathbb{Z}^n \times \mathbb{Z}_{+*}\}_{1 \leqslant i \leqslant m}$ where:

- $(\vec{v_i}, w_i)$ is called **elementary displacement**,
- $\vec{v_i}$ is a non null **vector** and
- w_i is a positive **weights** associated to $\vec{v_i}$.

Chamfer distance in $\mathbb{Z}^n$

$$d_{\mathcal{M}}(p,q) = \min\left\{ \sum \lambda_i w_i \ : \ \sum \lambda_i \vec{v_i} = \vec{pq}\ ,\ 1 \leqslant i \leqslant m,\ \lambda_i \in \mathbb{Z}_+ \right\}\ . \quad (1)$$

Chamfer distance in $\mathbb{R}^n$

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Chamfer norms in $\mathbb{Z}^n$

Chamfer norms in $\mathbb{R}^n$

Chamfer norms in $\mathbb{R}^n$ = polyhedral gauges for their unit balls.

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In $\mathbb{Z}^n$, chamfer norms = gauss discretization of chamfer norms in $\mathbb{R}^n$

Corollary

There is no metric basis in $\mathbb{Z}^n$ for chamfer norms.

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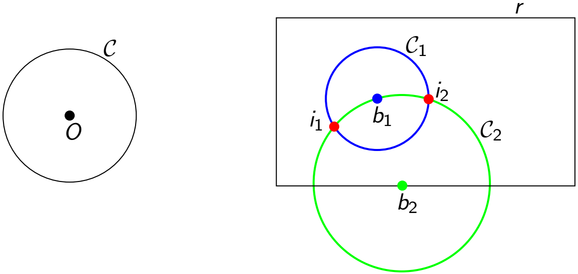
Basis points on the frontier

Lemma

If the metric dimension of a gauge is 2 *then both points of the bases are placed on the frontier of the rectangle.*



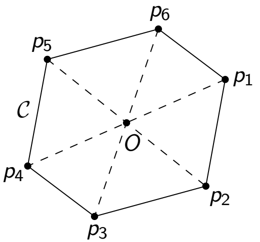
Proof illustration

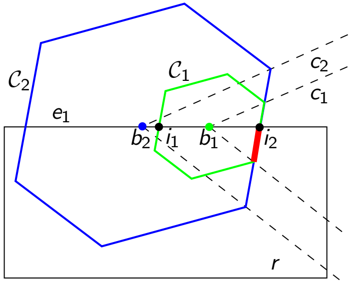
The red points i_1 and i_2 have the same base coordinates.
$(\{b_1\}, \{b_2\})$ is not a resolving set.

Basis points on the frontier

Lemma

If the metric dimension of a **polyhedral** *gauge is* 2 *and the points* b_1 *and* b_2 *are on the same edge* e*, then they are both corners.*





Proof illustration

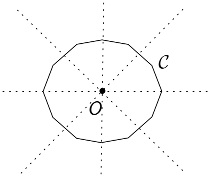
The points on the red bolded line have the same basse coordinates.
$(\{b_1\}, \{b_2\})$ is not a resolving set.

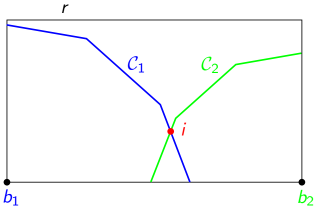
Grid symmetric gauges

Lemma

Suppose that $\gamma_{\mathcal{C}}$ is a **grid-symmetric**[a] *gauge. If $\mathcal{C}$ does not contain any vertical nor horizontal facet, then the metric dimension of $(r, d_{\mathcal{C}})$ is 2.*

[a] Axis and diagonal symmetries





Proof idea

Curves $\mathcal{C}_1$ and $\mathcal{C}_2$ are strictly monotonic. Intersection is at most a single point i.
$(\{b_1\}, \{b_2\})$ is a metric basis for r.

Metric dimension of Minkowski distances in a rectangle

Definition

The p-Minkowski distance is given by $d_p = \sqrt[p]{\sum_{i=0}^{n} |x_i|^p}$

Corollary

The metric dimension in a rectangle for any finite Minkowski distances (except d_∞) is 2.

Metric dimension of Minkowski distances in a rectangle

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The sole distance having a vertical or horizontal face if d_∞, thus

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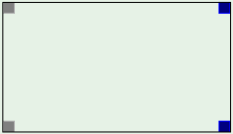
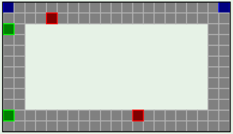
Candidate basis points

Examples for a 20 by 12 rectangle

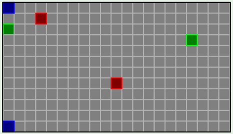
For the d_1 distance (Chamfer mask $\langle 1, 2 \rangle$)



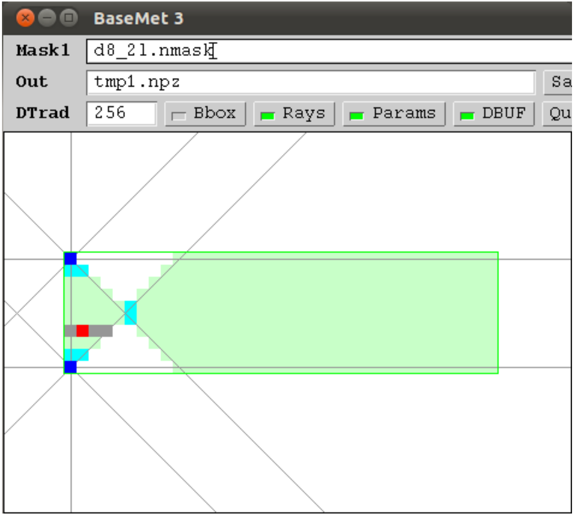


For the Chamfer mask $\langle 3, 4 \rangle$

For the Chamfer mask $\langle 5, 7, 11 \rangle$



d_∞ gauge in a rectangle



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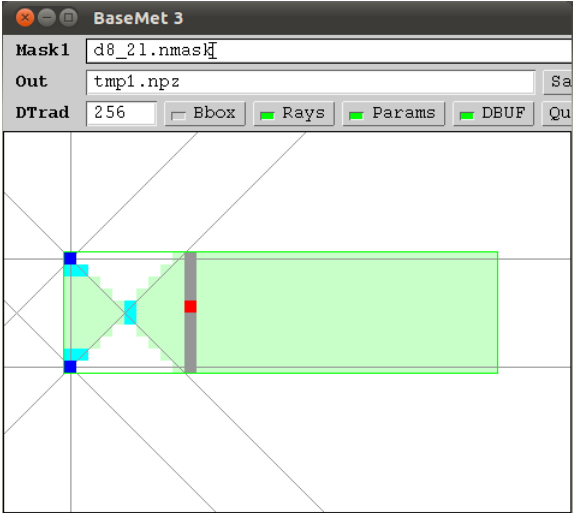

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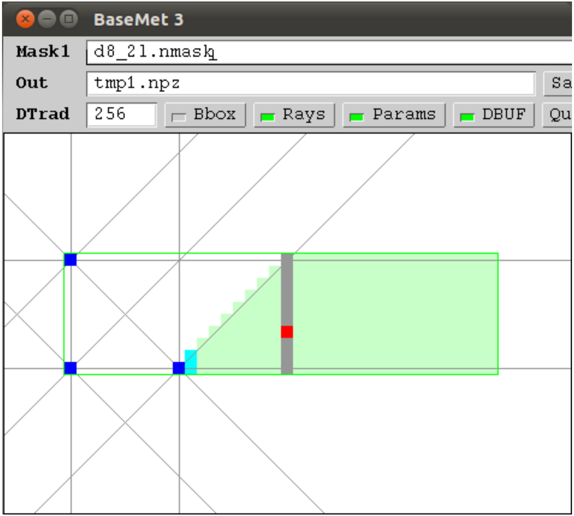

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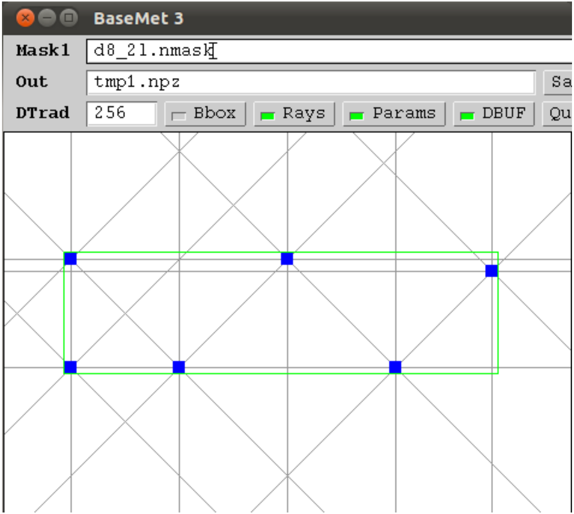


d_∞ gauge in a rectangle

Conjecture

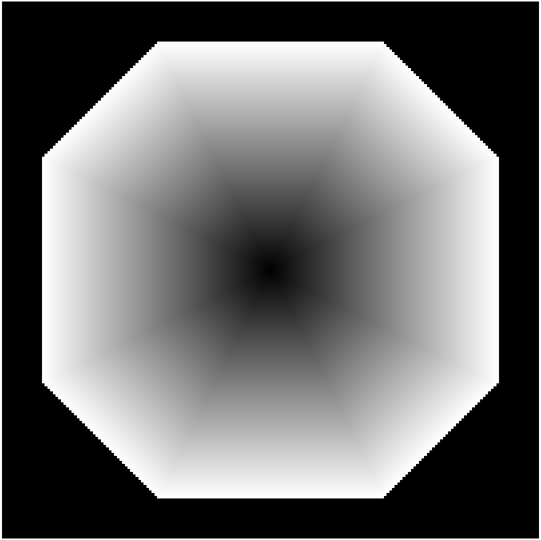
For a $a \times b$ rectangle, we have conjectured the metric dimension by :

- $b = 1 \quad \Longrightarrow \mathbf{1}$
- $b = a > 1 \Longrightarrow \mathbf{2}$
- $a > b > 1 \Longrightarrow (\mathbf{2} + \lfloor \frac{\mathbf{a-2}}{\mathbf{b-1}} \rfloor)$

Extreme case

If the rectangle have 2 as height or width, the metric basis will be half points of the rectangle.

Chamfer norm $\langle 3, 4, 6 \rangle$ in a rectangle



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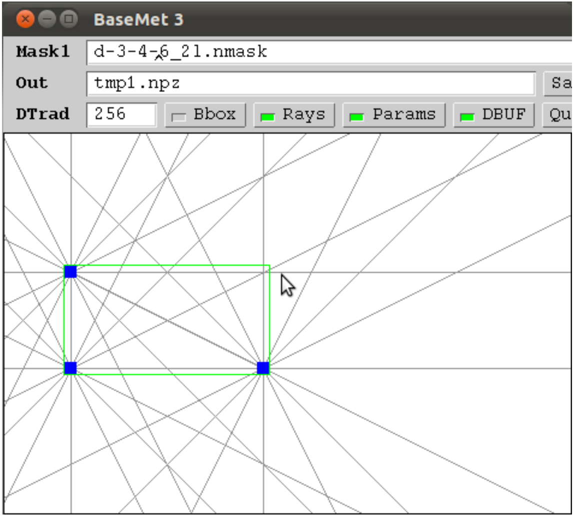

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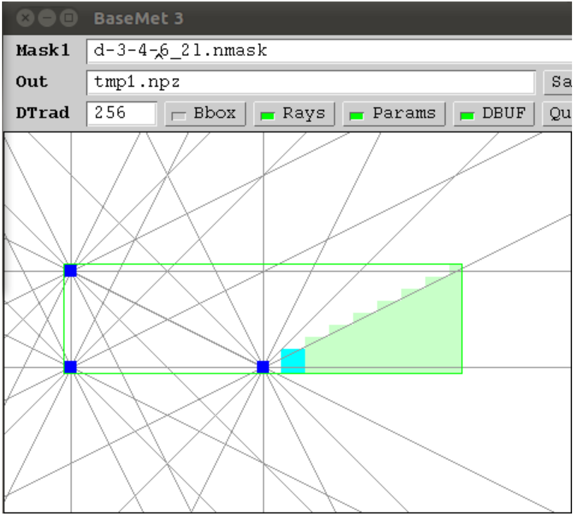
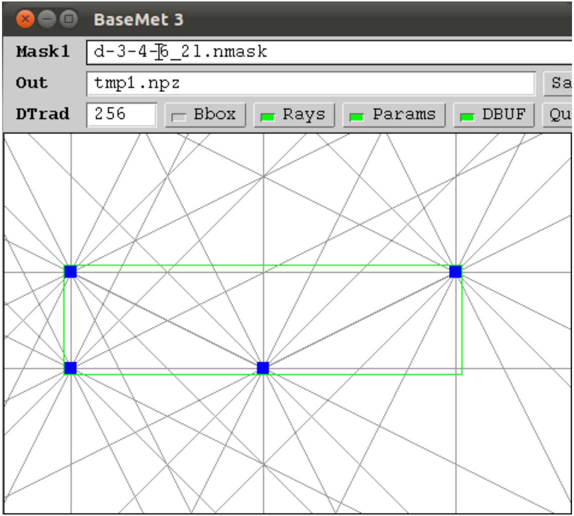


Chamfer norm $\langle 3, 4, 6 \rangle$ in a rectangle



Conclusion

Our contributions :

- polyhedral gauges do not have finites bases in $\mathbb{R}^n$,
- the same holds for rational gauges in $\mathbb{Z}^n$ (Chamfer Norms),
- some $\mathbb{R}^n$ properties disappear in $\mathbb{Z}^n$,
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Future prospects :

- Higher dimensions,
- Rectangles, convex or non-convex polyhedrons with direct and geodesic distances.
- Linked problems of forcing subsets and partition dimensions.