Delaunay properties of digital straight segments

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Outline

Definitions: patterns and Delaunay triangulation

Observation: Delaunay triangulation of patterns?

Characterization: proof

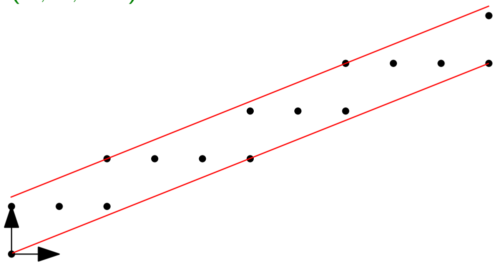
Conclusion: new output-sensitive algorithms

Digital straight line (DSL)

Standard DSL

The points $(x, y) \in \mathbb{Z}^2$ verifying $\mu \leq ax - by < \mu + |a| + |b|$ belong to the standard DSL $D(a, b, \mu)$ of slope $\frac{a}{b}$ and intercept μ ($a, b, \mu \in \mathbb{Z}$ and $pgcd(a, b) = 1$).

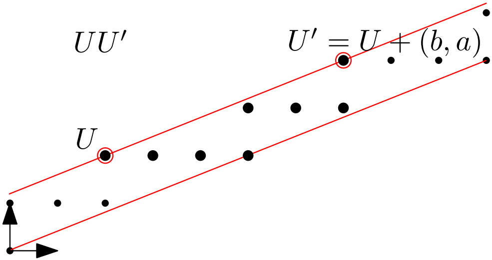
Example: $D(2, 5, -6)$



Pattern

- a pattern is a connected subset of a DSL between two consecutive upper leaning points

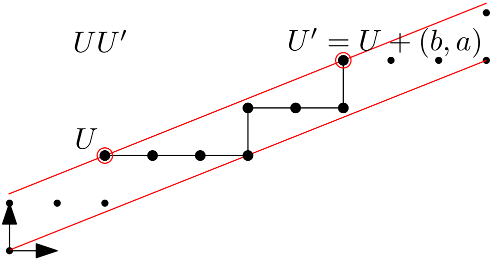
Example: pattern *UU′*



Pattern

- a pattern is a connected subset of a DSL between two consecutive upper leaning points
- its staircase representation is the polygonal line linking the points in order

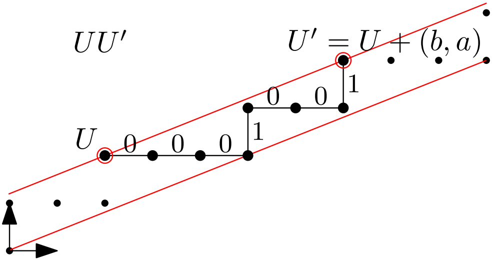
Example: pattern *UU′*



Pattern

- a pattern is a connected subset of a DSL between two consecutive upper leaning points
- its staircase representation is the polygonal line linking the points in order
- its chain code is a Christoffel word

Example: pattern *UU′*



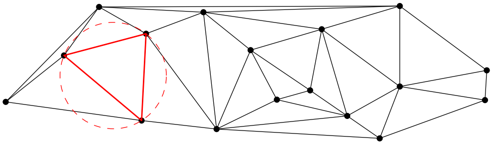
Delaunay triangulation

Triangulation of a finite set of points $\mathcal{S}$

Partition of the convex hull of $\mathcal{S}$ into triangular facets, whose vertices are points of $\mathcal{S}$.

Delaunay condition

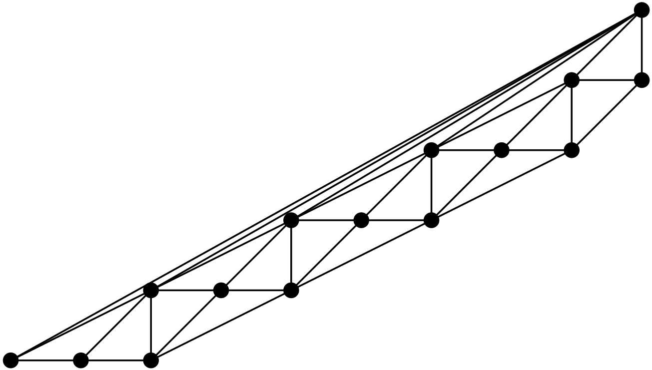
The interior of the circumcircle of each triangular facet does not contain any point of $\mathcal{S}$.



always exists and is unique (without 4 cocircular points)

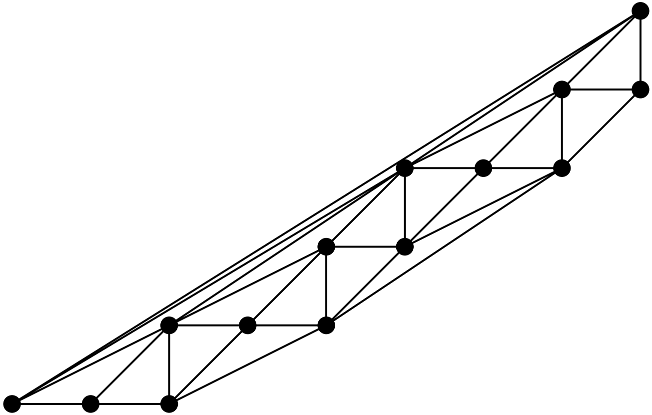
Delaunay triangulation of patterns

Pattern of slope $5/9$



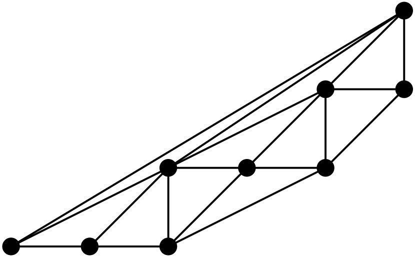
Delaunay triangulation of patterns

Pattern of slope $5/8$



Delaunay triangulation of patterns

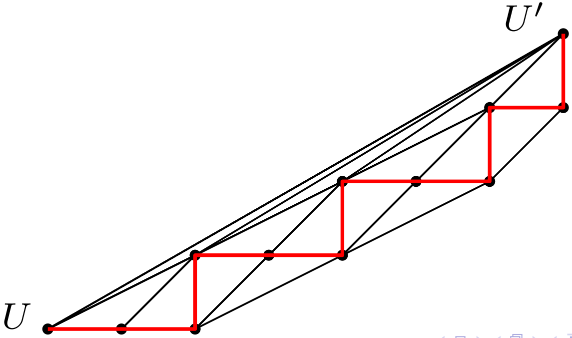
Pattern of slope $2/5$



Three remarks

1. the Delaunay triangulation of *UU′* contains the staircase representation of *UU′*.

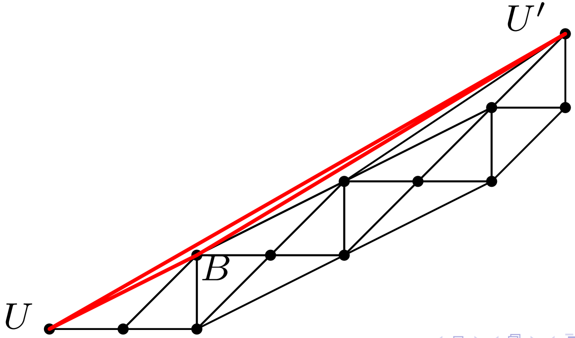
Pattern of slope $4/7$



Three remarks

1. the Delaunay triangulation of UU' contains the staircase representation of UU'.
2. U, U' and the closest point of UU' to $[UU']$ (Bezout point) define a facet.

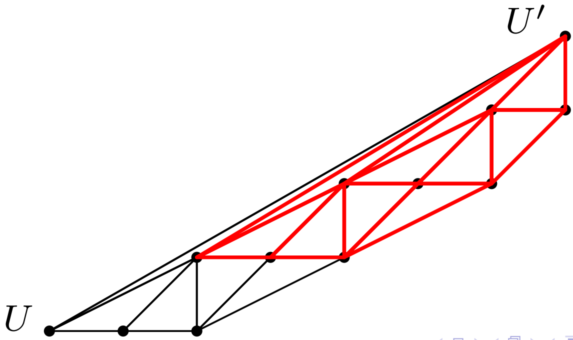
Pattern of slope $4/7$



Three remarks

1. the Delaunay triangulation of UU' contains the staircase representation of UU'.
2. U, U' and the closest point of UU' to $[UU']$ (Bezout point) define a facet.
3. the Delaunay triangulation of some patterns contains the Delaunay triangulation of subpatterns.

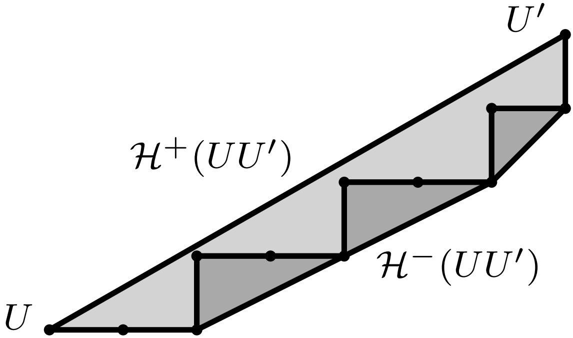
Pattern of slope $4/7$



Dividing the triangulation (remark 1)

- The convex hull of UU' is divided into a upper part $\mathcal{H}^+(UU')$ and a lower part $\mathcal{H}^-(UU')$.

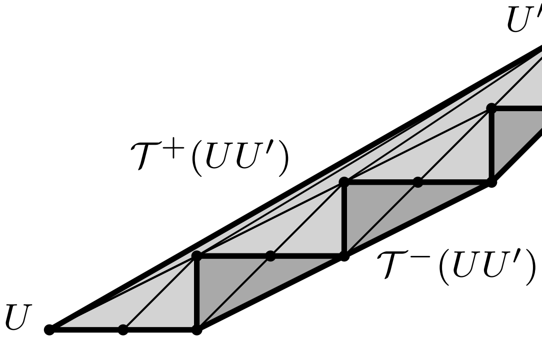
Pattern of slope $4/7$



Dividing the triangulation (remark 1)

- The convex hull of UU' is divided into a upper part $\mathcal{H}^+(UU')$ and a lower part $\mathcal{H}^-(UU')$.
- The Delaunay triangulation of UU' is divided into a upper part $\mathcal{T}^+(UU')$ and a lower part $\mathcal{T}^-(UU')$.

Pattern of slope $4/7$



Main facet of a pattern (remark 2)

Main facet = triangle UBU'

Let B the Bezout point of UU' and let

- $[q_0; \ldots, q_i, \ldots, q_n]$ (with $q_n > 1$) be the quotients and
- $(b_0, a_0), \ldots, (b_i, a_i), \ldots, (b_n, a_n)$ be the convergent vectors

of the continued fraction expansion of $\frac{a}{b}$.

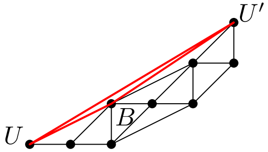
$$\overrightarrow{UU'} = \overrightarrow{UB} + \overrightarrow{BU'} = (b_n, a_n) + \big((q_n - 1)(b_n, a_n) + (b_{n-1}, a_{n-1})\big)$$

Equivalent to the *splitting formula* [Voss, 1993] only expressed in terms of quotients.

Set of facets of a pattern (remark 3)

UB and *BU′* are both patterns
their chain code are Christoffel words

- other facets defined by induction
  - geometrical characterization (Bezout point)
  - combinatorial characterization (splitting formula)



0 0 1 | 0 0 1 0 1

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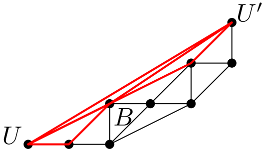

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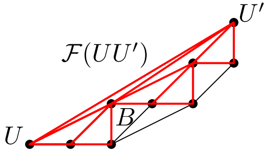


  - combinatorial characterization
    (splitting formula)

0 | 0|1|0 | 0|1|0|1

Main result

Theorem

The facets $\mathcal{F}(UU')$ of the pattern UU' is a triangulation of $\mathcal{H}^+(UU')$ such that each facet has points of UU' as vertices and satisfies the Delaunay property, i.e. $\mathcal{F}(UU') = \mathcal{T}^+(UU')$.

the (upper part of the) Delaunay triangulation of a pattern is characterized by the continued fraction expansion of its slope

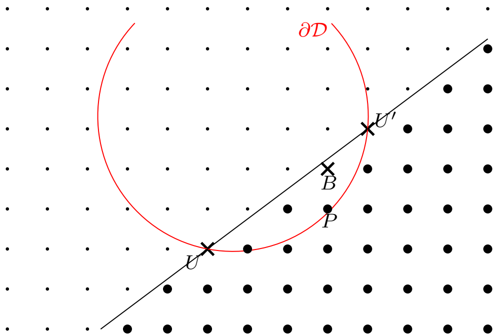
Sketch of the proof

We have to show that:

1. the set of facets $\mathcal{F}(UU')$ is a triangulation of $\mathcal{H}^+(UU')$ (easy part)
2. the interior of the circumcircle of each facet of $\mathcal{F}(UU')$ does not contain any point of UU'
   (let us focus on that part)

Lemma 1

Let $\mathcal{D}$ be a disk whose boundary passes through U and U' and whose center is located above (UU'). The interior of $\mathcal{D}$ contains a lattice point below or on (UU') if and only if it contains (at least) B, the Bezout point of UU'.



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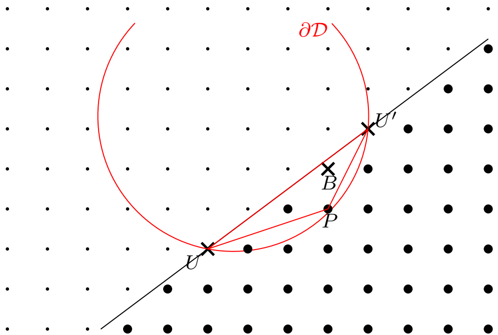

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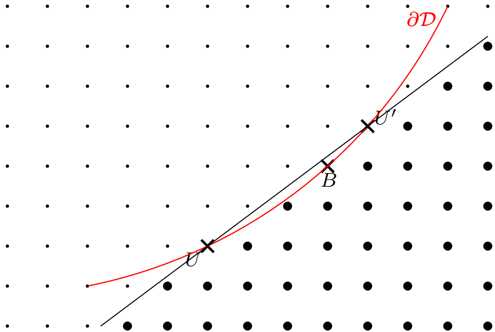
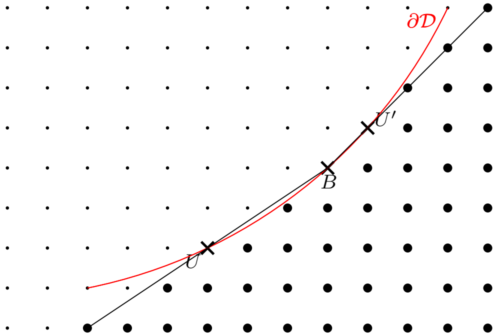


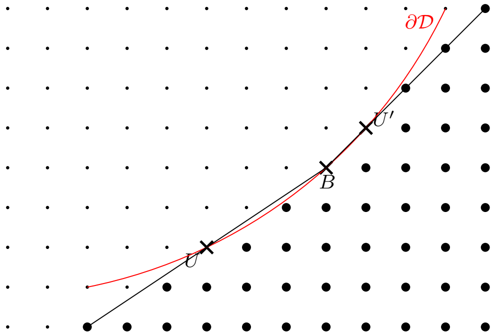
Lemma 2

Let $\mathcal{D}$ be a disk whose boundary $\partial\mathcal{D}$ is the circumcircle of *UBU′*. The interior of $\mathcal{D}$ contains none of the *background points* of *UU′* (lattice points below straight lines (*UB*) or (*BU′*)).



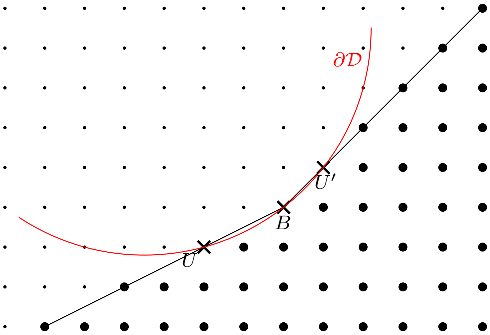
Applying lemma 2 by induction over all the facets

The background points of UU' (which contains UU') are contained in the background points of UB (and BU').



Applying lemma 2 by induction over all the facets

The background points of *UU′* (which contains *UU′*) are contained in the background points of *UB* (and *BU′*).



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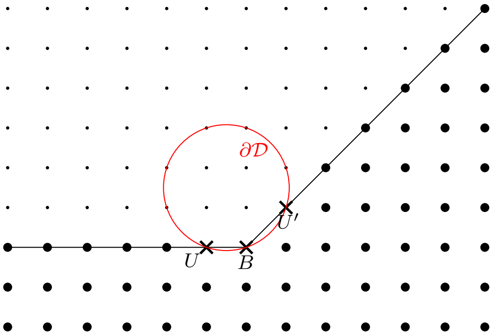
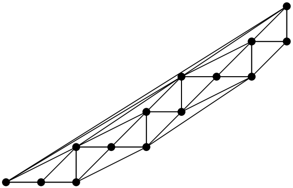


Outline

Delaunay triangulation computation

- Pattern

pattern of slope $8/5$



Delaunay triangulation computation

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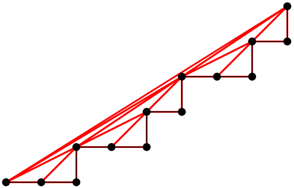

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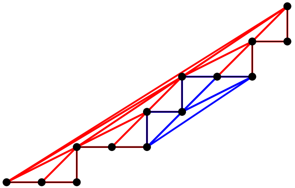

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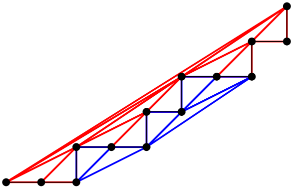

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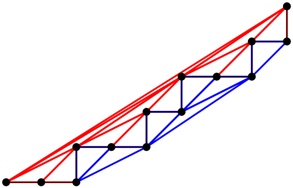
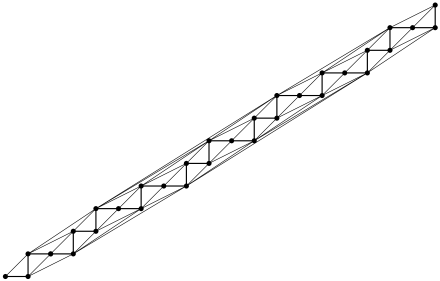


Delaunay triangulation computation

- Pattern
- DSS

DSS of slope $8/5$



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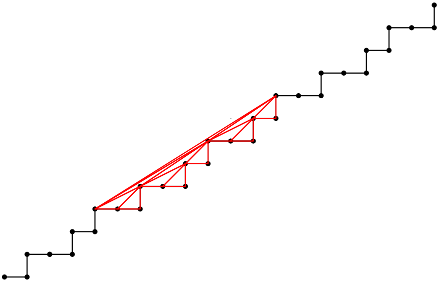

Delaunay triangulation computation

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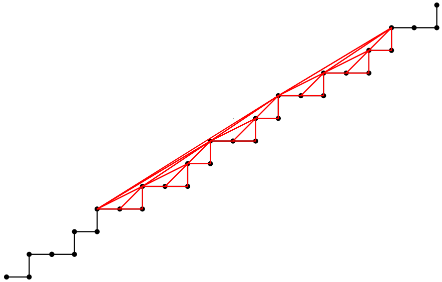
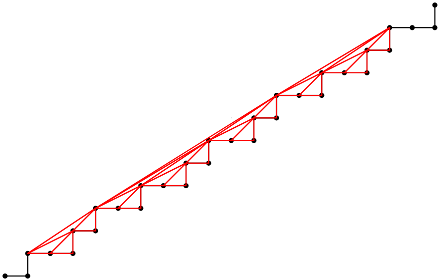
DSS of slope 8/5



Delaunay triangulation computation

- Pattern
- DSS

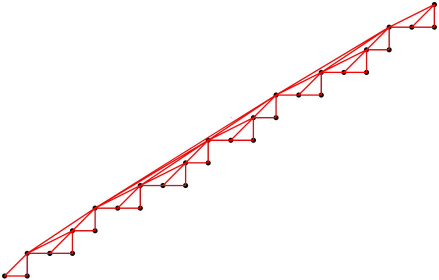
DSS of slope $8/5$



Delaunay triangulation computation

- Pattern
- DSS

DSS of slope $8/5$



Delaunay triangulation computation

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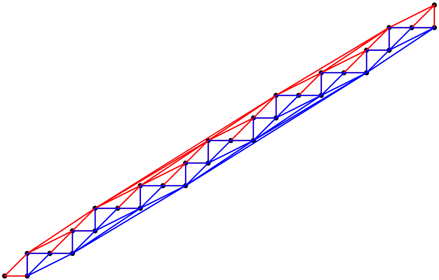
- DSS

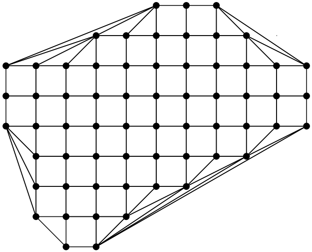
DSS of slope $8/5$



Delaunay triangulation computation

- Pattern
- DSS
- Convex digital object

Convex digital object



Perspectives

New linear-time and output-sensitive algorithms to compute geometrical structures from specific sets of lattice points.

- study more geometrical structures:
  - Delaunay triangulation, Voronoï diagram
  - α-hull, α-shape
  - medial axis, skeleton
- study other sets:
  - patterns, DSSs
  - convex digital objects
  - two consecutive maximal segments
  - convex digital boundaries

C'est fini!